SOLUTION OF STATISTICAL PROBLEMS IN ELASTICITY THEORY IN THE SINGULAR APPROXIMATION

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A method of calculating elastic fields and effective moduli of microheterogeneous solids is developed in the random field theory. The solution is obtained in the form of an operator series, each term of which is constructed on the basis of the regular component of the second derivative tensor of the equilibrium Green function. The zeroth approximation of such a series consists of the local part of the interaction between inhomogeneity grains. The possibilities of the method are illustrated on the example of an isotropic mixture of two isotropic components.

One of the main problems of the elasticity theory of microheterogeneous bodies is to find the tensors of effective elastic moduli and of elastic fields of statistically homogeneous media. In solving this problem both classical methods of elasticity theory [1-3] and methods of random field theory [4-9] may be used.

The presence of interactions between grain inhomogeneities puts the problem mentioned in the class of well-known multiparticle problems, whose exact solution may be found only in the simplest cases. In this connection it is of interest to develop approximate methods of calculating elastic properties of inhomogeneous materials [1-10], using the characteristics of the simplified problem with the purpose of solving it to the end.

Three methods provide the highest accuracy, the self-consistent [2, 10], the variational [3], and the random field method [4-9]. The latter underwent several modifications, of which the Bolotin–Kroner [6, 9] model allows the most transparent physical interpretation.

Despite the difference between the methods mentioned, the approximate effective elastic moduli, obtained by them, can be reduced to an identical analytic form. This fact underlies the idea of their being similar in principle.

In this paper we develop a unified approach to solving the problem of describing inhomogeneous elastic media, satisfying equilibrium equations. This method can be extended to solve other problems without particular difficulty [5].

1. Let the statistical homogeneity of the infinite medium under consideration be characterized by an elastic moduli tensor λ_{ijkl} (r). The field of this tensor will obviously possess a constant average value and a random component. We introduce besides a reference field whose elastic properties are characterized by some homogeneous tensor λ°_{ijkl}.

The fields u_i and u_i°, corresponding to both elastic moduli tensors, satisfy the equations

$$L_{ik}u_k = -f_i, \quad L_{ik} = \nabla_j\lambda_{ijkl}\nabla_l, \quad \overset{\circ}{L}_{ik}\, u_k^{\circ} = -f_i, \quad L_{ik}^{\circ} = \nabla_j\lambda_{ijkl}^{\circ}\nabla_l \tag{1.1}$$

where f_i is the vector density of bulk forces.

The problem consists of finding the tensors of deformation $\varepsilon_{ij} = {}^1\!/_2(\nabla_i u_j + \nabla_j u_i) = u_{(i,\,j)}$ and of effective elastic moduli λ^*_{ijkl}. The latter determines the average deformation $\langle \varepsilon_{ij} \rangle = \langle u_{(i,\,j)} \rangle$ by the equation

$$L^*_{ik}\langle u_k \rangle = -f_i, \qquad L^*_{ik} = \nabla_j\lambda^*_{ijkl}\nabla_l \tag{1.2}$$

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The angular brackets denote ensemble averaging, which for ergodic fields coincides with volume averaging.

Denoting by primes the excess with respect to the reference body functions, we find from (1.1)

$$\overset{\circ}{L}_{ik}u_k' = -L_{ik}'u_k, \quad L_{ik}' = L_{ik} - \overset{\circ}{L}_{ik}, \quad u_k' = u_k - u_k^\circ \tag{1.3}$$

The solution of (1.3) by means of the Green tensor G_{ik} of the operator L_{ik} is of the form

$$u_i' = G_{ik} * L_{kl}' u_l \tag{1.4}$$

where the star denotes the integral convolution operation.

For the excess deformation we have from (1.4)

$$\varepsilon_{ij}' = u_{(i,\,j)}' = G_{k)(i,\,j)} * (_l\lambda_{klmn}'\varepsilon_{mn} \tag{1.5}$$

Using the ideas of the generalized functions theory, we write down the second derivative of the Green tensor as a sum of singular and regular parts [4]

$$G_{ik,\,jl} = G_{ik,\,jl}^S + G_{ik,\,jl}^f \tag{1.6}$$

We introduce the tensor g_{ijkl} and the integral operator p_{ijkl}

$$g_{ijkl}F = G_{i)\,(k,\,l)\,(j}^S * F, \qquad p_{ijkl}F = G_{i)\,(k,\,l)\,(j}^f * F \tag{1.7}$$

where F is an arbitrary function.

The first of relations (1.7) is possible due to the fact that the coordinate part of $G_{ik,\,jl}^S$ is δ (r). Besides, in what follows we omit the tensor indices, regarding second-rank tensors as vectors and fourth-rank tensors as square matrices in six-dimensional space [11].

It is easily seen that by means of (1.7) Eq. (1.5) can be transformed to the form

$$e = \varepsilon^\circ + ple, \quad e \equiv (1 - g\lambda')\,\varepsilon \tag{1.8}$$

The tensor l is defined by

$$l^{-1} = (\lambda - \lambda_0)^{-1} - g \tag{1.9}$$

The advantage of (1.8) over (1.5) is that its main part, related to the singular derivative of the Green tensor, is separated from the term describing the nonlocal part of the interactions between grain inhomogeneities.

Eqs. (1.8) and (1.9) allow us to determine the tensors of effective elastic moduli λ_* and of the field deformation ε. Indeed, from (1.8) and (1.9) we have

$$le = (\lambda - \lambda_0)\,\varepsilon \tag{1.10}$$

which after averaging gives

$$l_*\langle e\rangle = \langle le\rangle = \langle(\lambda - \lambda_0)\varepsilon\rangle = (\lambda_* - \lambda_0)\,\langle\varepsilon\rangle \tag{1.11}$$

In view of the analytic identity of (1.10) and (1.11), and accounting for the relation between $\langle e \rangle$ and $\langle \varepsilon \rangle$ following from definition (1.8), the connection between the effective tensors l_* and λ_* is described by Eq. (1.9).

It is not hard to see from (1.8) that the field e can be represented in the form

$$e = (1 - pl)^{-1}\,\varepsilon_0 = \Sigma\,(pl)^n\varepsilon_0 \tag{1.12}$$

Averaging (1.12), we obtain

$$\langle e\rangle = \langle(1 - pl)^{-1}\rangle\,\varepsilon_0 \tag{1.13}$$

Eliminating the field ε_0 from (1.12) and (1.13), we find

$$e = (1 - pl)^{-1}\,\langle(1 - pl)^{-1}\rangle^{-1}\,\langle e\rangle \tag{1.14}$$

Substituting (1.14) in (1.11) gives

$$l_* = \langle l\,(1 - pl)^{-1}\rangle\,\langle(1 - pl)^{-1}\rangle^{-1} \tag{1.15}$$

2. Despite their formal simplicity, the solutions (1.14) and (1.15) represent operator series [4, 5], requiring for their evaluation the values of multipoint moment functions of elastic constants. Since the mathematical difficulty in obtaining high-order approximations is considerable, and modelling does not yet provide adequate representation of microheterogeneous media, it is customary to restrict the discussion to lowest order approximations in operator series of type (1.14) and (1.15).

Consider the singular approximation which consists of neglecting the interactions between inhomogeneity elements related with the operator p, which takes into account the deviation of the field ε at a given point from its average over the grain.

This corresponds to the zeroth approximation in Eqs. (1.14) and (1.15), having the form

$$e_S = \langle e \rangle, \ l_S = \langle l \rangle \tag{2.1}$$

where the index S denotes that the field e and the tensor l_* are evaluated in the singular approximation. The field ε_S is easily found from definition (1.8)

$$\varepsilon_S = (1 - g\lambda')^{-1} e_S \tag{2.2}$$

Averaging (2.2) and eliminating e_S, we obtain

$$\varepsilon_S = (1 - g\lambda')^{-1} \langle (1 - g\lambda')^{-1} \rangle^{-1} \langle \varepsilon \rangle \tag{2.3}$$

By means of Eq. (2.3) the effective elastic moduli tensor λ_* is determined by the equation

$$\langle \lambda\varepsilon \rangle = \lambda_* \langle \varepsilon \rangle \tag{2.4}$$

and has the form

$$\lambda_S = \langle \lambda (1 - g\lambda')^{-1} \rangle \langle (1 - g\lambda')^{-1} \rangle^{-1} \tag{2.5}$$

Expression (2.5) can also be obtained directly from (1.9), replacing l and λ by l_S and λ_S.

Introducing the vectorial tensor b_0, satisfying the relation

$$g (\lambda_0 + b_0) = -1 \tag{2.6}$$

Eqs. (2.3) and (2.5) are simplified and are reduced to the form

$$\varepsilon_S = (\lambda + b_0)^{-1} \langle (\lambda + b_0)^{-1} \rangle^{-1} \langle \varepsilon \rangle = A_S \langle \varepsilon \rangle, \ (\lambda_S + b_0)^{-1} = \langle (\lambda + b_0)^{-1} \rangle \tag{2.7}$$

Since generalized functions, and, consequently, their derivatives, are domain functions restricting the region of coordinates [12], the tensors g and b_0 must depend on the shapes of the surfaces of these domains. For the domain mentioned we choose the effective grain, over which the ensemble averaging of the inhomogeneity grain should be performed.

Let the effective grain be an ellipsoid with major axes a_1, a_2, a_3 in a Cartesian coordinate system, and let the tensor λ_0 be isotropic. The tensor g will then have the crystal symmetry of an orthorhombic system [11] and will be written in the form

$$-\mu_0 g_{ijkl} = \delta_{i)(k} J_{l)(j} - \chi J_{ijkl}, \qquad \chi = \frac{\lambda_0 + \mu_0}{\lambda_0 + 2\mu_0} \tag{2.8}$$

Here λ_0 and μ_0 are the Lame constants, determining the tensor λ_{ijkl}, and the components of the tensors J_{ij} and J_{ijkl} are defined by the integral [7]

$$J_{11} = \frac{1}{4\pi a_1^2} \int \frac{n_1^2 d\Omega}{\Sigma n_i^2 a_i^{-2}}, \qquad n_i = r_i / r$$

$$J_{1111} = \frac{1}{4\pi a_1^4} \int \frac{n_1^4 d\Omega}{(\Sigma n_i^2 a_i^{-2})^2}, \quad J_{1122} = \frac{1}{4\pi a_1^2 a_2^2} \int \frac{n_1^2 n_2^2 d\Omega}{(\Sigma n_i^2 a_i^{-2})^2} \tag{2.9}$$

where $d\Omega$ is the solid angle element.

The remaining components of the tensors J_{ij} and J_{ijkl} are obtained by corresponding index permutations. In the case under consideration the depolarization tensor J_{ij} can be expressed in terms of elliptic integrals [13], and the tensor J_{ijkl} in terms of their derivatives.

Eqs. (2.7)-(2.9) are the singular approximation solutions of the problem of finding the deformation field and the effective elastic moduli of a microheterogeneous medium. The solution obtained allows the checking of various structures: mixtures whose components can possess arbitrary symmetry, multiphase polycrystals, mechanical and orientational textures. In the given approximation, however, no distinction is made between matrix and statistical mixture, which is related to neglecting elastic field inhomogeneities inside grains. The latter is, obviously, very sensitive to the distribution of elastic fields in adjacent grains.

3. As an example we consider nontexturized, mechanical mixtures of two isotropic components. In this case the effective grain is of spherical shape, and the tensors J_{ij} and J_{ijkl} equal

$$J_{ij} = {}^1/_3\delta_{ij}, \quad J_{ijkl} = {}^1/_3 V_{ijkl} + {}^2/_{15} D_{ijkl} \qquad (3.1)$$
$$3V_{ijkl} = \delta_{ij}\delta_{kl}, \quad V_{ijkl} + D_{ijkl} = \delta_{i(k}\delta_{l)j}$$

Substituting (3.1) into (2.8) and using (2.6), we find an explicit form of the tensor $\overset{\circ}{b}_{ijkl}$

$$\overset{\circ}{b}_{ijkl} = 3b_0 V_{ijkl} + 2d_0 D_{ijkl} \qquad (3.2)$$
$$3b_0 = 4\mu_0, \quad d_0 = \frac{\mu_0}{6}\frac{9K_0 + 8\mu_0}{K_0 + 2\mu_0}, \quad 3K_0 = 3\lambda_0 + 2\mu_0$$

Since the symmetry of the tensors λ_{ijkl}, $\overset{\circ}{\lambda}_{ijkl}$, and $\overset{\circ}{b}_{ijkl}$ is the same, Eq. (2.7) is easily calculated, and together with (3.2) it leads to the result

$$A^S_{ijkl} = \frac{K_S + b_0}{K + b_0} V_{ijkl} + \frac{\mu_S + d_0}{\mu + d_0} D_{ijkl} \qquad (3.3)$$
$$\lambda^S_{ijkl} = 3K_S V_{ijkl} + 2\mu_S D_{ijkl}$$

where K_S and μ_S are of the form

$$K_S = \langle K \rangle - \frac{D_K}{c_1 K_2 + c_2 K_1 + b_0}, \quad \mu_S = \langle \mu \rangle - \frac{D_\mu}{c_1\mu_2 + c_2\mu_1 + d_0}$$
$$D_x = c_1 c_2 (x_1 - x_2)^2 \qquad (3.4)$$

and the indices 1 and 2 denote the number of the component.

Since the quantities b_0 and d_0 are determined from the Lame constants λ_0 and μ_0 of the reference body, expressions (3.3) and (3.4) are also functions of λ_0 and μ_0. Assigning to them different values, we obtain the results of the approximate methods discussed above. We restrict the discussion to the analysis of Eq. (3.4).

Let K_0 and μ_0 obtain the following values:

a) $K_0 = K_S$, $\mu_0 = \mu_S$; b) $K_0^+ = K_2$, $\mu_0^+ = \mu_2$, $K_0^- = K_1$, $\mu_0^- = \mu_1$, under the conditions $K_1 < K_2$, $\mu_1 < \mu_2$; c) $K^u_0 = \langle K \rangle$, $\mu^u_0 = \langle \mu \rangle$; $K_0^l = \langle 1/K \rangle^{-1}$, $\mu_0^l = \langle 1/\mu \rangle^{-1}$.

Substituting them in (3.4) gives the effective elastic moduli of the self-consistent approximation [2, 10] in case (a), the Hashin–Shtrikman bounds obtained by means of variational principles [3] in case (b), and the approximate values found by the random field theory [4, 5, 8] in case (c).

Similar results can also be obtained in considering more complicated types of microheterogeneous media.

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